# Pythagorean and Heronian triangles

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The basic Pythagorean theorem for right-angled triangles is well known in mathematical terms as

$$a^2 + b^2 = c^2 \tag{1}$$

where $a$, $b$, $c$ are the lengths of the sides of the triangle with $c$ as the hypotenuse. It follows that if

$$a^2 + b^2 \neq c^2$$

for a triangle then it is not right-angled. Some simple deductions about the type of triangle can then be made for the two cases $a^2 + b^2 < c^2$ and $a^2 + b^2 > c^2$.

We also note that if a triangle is right angled, it does not necessarily have all its sides rational. A simple example is $a = 1$, $b = \sqrt{3}$, $c = 2$. However, when $a$, $b$, $c$ are all integers and obey equation (1), they are referred to as a Pythagorean triple [$a$, $b$, c]. In addition, when $a$, $b$, $c$ have no common factor other than 1, they form a primitive Pythagorean triple. Thus, [3, 4, 5] is a primitive Pythagorean triple, but [6, 8, 10] is only a Pythagorean triple.

DeMestre (1995) showed how to generate a table of Pythagorean triples using [$a$, $b$, $b$ + 1] and then [$a$, $b$, $b$ + 2]. The first table produces [3, 4, 5], [5, 12, 13], [7, 24, 25] as the first three entries, with [8, 15, 17] and [10, 24, 26] as the first two entries of the second table. Ask your class to find the first 10 entries of each. Are there any triples of the form [$a$, $b$, $b$ + 3]?

One interesting method to generate Pythagorean triples is to start with two fractional numbers whose product is 2. Add 2 to each factor, multiply both answers by the lowest common multiple of their denominators, and the resulting two numbers are the first two in a Pythagorean triple. For example,

$$\frac{4}{3} \times \frac{3}{2} = 2$$

Consider $\frac{4}{3} + 2 = \frac{10}{3}$ and $\frac{3}{2} + 2 = \frac{7}{2}$.

Multiply by the lowest common multiple 6, yielding 20 and 21, which are the first two of the triple [20, 21, 29].

The proof of this fascinating method for generating Pythagorean triples is based on the two numbers $z$ and $2/z$. These yield eventually $z(2 + z)$ and $(2z + 2)$, giving rise to the identity:

$$(z^2 + 2z)^2 + (2z + 2)^2 = (z^2 + 2z + 2)^2$$

This formula generates many Pythagorean triples, but not all of them. However the two-variable identity:

$$(m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2$$

does generate all the Pythagorean triples if $m$ and $n$ are integers such that $m > n \geq 1$. (The first identity is seen to be a special case of this one if we put $m = z + 1$ and $n = 1$.)

One property of right-angled triangles based on Pythagorean triples is that their areas are always integers. This leads to the question: are there are any triangles, other than right-angled ones, which also have integer sides and areas? The answer is: yes. These triangles will be referred to as Heronian triangles (Cheney, 1929), because they depend on the rationalisation of the formula for the area of a triangle given by Heron (or Hero), a first century Egyptian engineer from Alexandria. If $A$, $B$, $C$ represent the lengths of the sides of a triangle, then its area is:

$$\sqrt{S(S-A)(S-B)(S-C)}$$

where $$S = \frac{A+B+C}{2}$$

This formula was actually discovered by Archimedes, but bears Heron's name. Your students can attempt to derive this result by using any altitude of the triangle plus Pythagoras' theorem together with expanding backwards from Heron's result.

The previous construction of an altitude suggests how to form a triangle with integer sides and integer area: simply combine two right-angled triangles made up of suitably selected Pythagorean triples.

The easiest selection is to use two exactly equal triples. For example, [3, 4, 5] with [3, 4, 5] can produce the Heronian triangles {5, 5, 6} and {5, 5, 8} depending on which shorter side is chosen as the altitude. Both these triangles are isosceles Heronian triangles, and there are an infinite number of them.

However, Heronian triangles that are not isosceles can also be formed. Let these triangles be called Q-triangles. For example, [5, 12, 13] and [12, 16, 20] can be additively combined with 12 as the altitude to produce the Heronian triangle {13, 20, 21} with area 126 square units or combined by using subtraction to produce {11, 13, 20} with area 66 square units (see Figure 1).

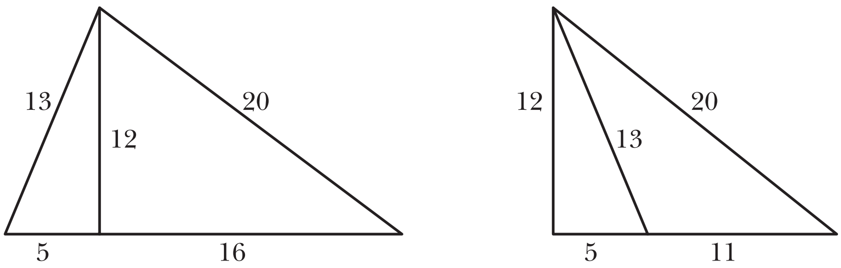


*Figure 1*

Thus it appears that Heronian triangles can be classified into three types: right-angled triangles, isosceles triangles and Q-triangles. Table 1 gives a list of Q-triangles for A < 9 where A < B < C.

See if your students can extend this table; then see if they can establish the following three theorems. They may need a hint such as: "You may need to use ideas from *similar triangles* and *modular arithmetic* and refer to the general identity above."

*Table 1. Q-triangles with A < 9*

| A | B | C | Area |
|---|---|---|---|
| 3 | 25 | 26 | 36 |
| 4 | 13 | 15 | 24 |
| 4 | 51 | 53 | 90 |
| 5 | 29 | 30 | 72 |
| 5 | 51 | 52 | 126 |
| 6 | 25 | 29 | 60 |
| 6 | 50 | 52 | 144 |
| 7 | 15 | 20 | 42 |
| 7 | 65 | 68 | 210 |
| 8 | 26 | 30 | 96 |
| 8 | 29 | 35 | 84 |

**Theorem 1**: All the altitudes of a Q-triangle are rational, but not necessarily integral.

**Theorem 2**: For any Q-triangle, each altitude divides the base into rational segments.

**Theorem 3**: The area of every Q-triangle has 6 as a factor.

Finally, here are two further problems to challenge your students.

1. Find two Q-triangles with all sides <100 units of the form {N, N+1, N+2}. Note that {3, 4, 5} is not Q.

2. Find five Heronian triangles such that their Area = A + B + C. (Hint: They are not Q.)

## References

Cheney, W. F. (1929). Heronian triangles. *American Mathematical Monthly, 36*, 22–28.

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